Recall: Two different types of surface integrals

- for functions
- for vector fields

In both case, we need a parametrization of surface S

$$\Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

image of Φ = $\Phi(D)$ = surface S

requirements for Φ:

it should be <u>onto</u> (every point of S is in image of Φ)

" " " (essentially) 1-1

i.e. for each (x,y,z) in S there should only be one (u,v) in D s.t. $\Phi(u,v) = (x,y,z)$

not always exactly the case.

example: 1 sphere of radius a

$$\Phi(\phi,\theta) = (a \sin\phi \cos\theta,\ a \sin\phi \sin\theta,\ a \cos\phi)$$

D given by $\quad 0 \le \phi \le \pi$
$\quad\quad\quad\quad\quad\quad 0 \le \theta \le 2\pi$

observe $\quad \Phi(\phi,0) = \Phi(\phi,2\pi) \quad$ (0 and 2π denote same angle!

this is the case only for the lines
$(\phi,0)$ and $(\phi,2\pi)$, $\quad 0 \le \phi \le \pi$

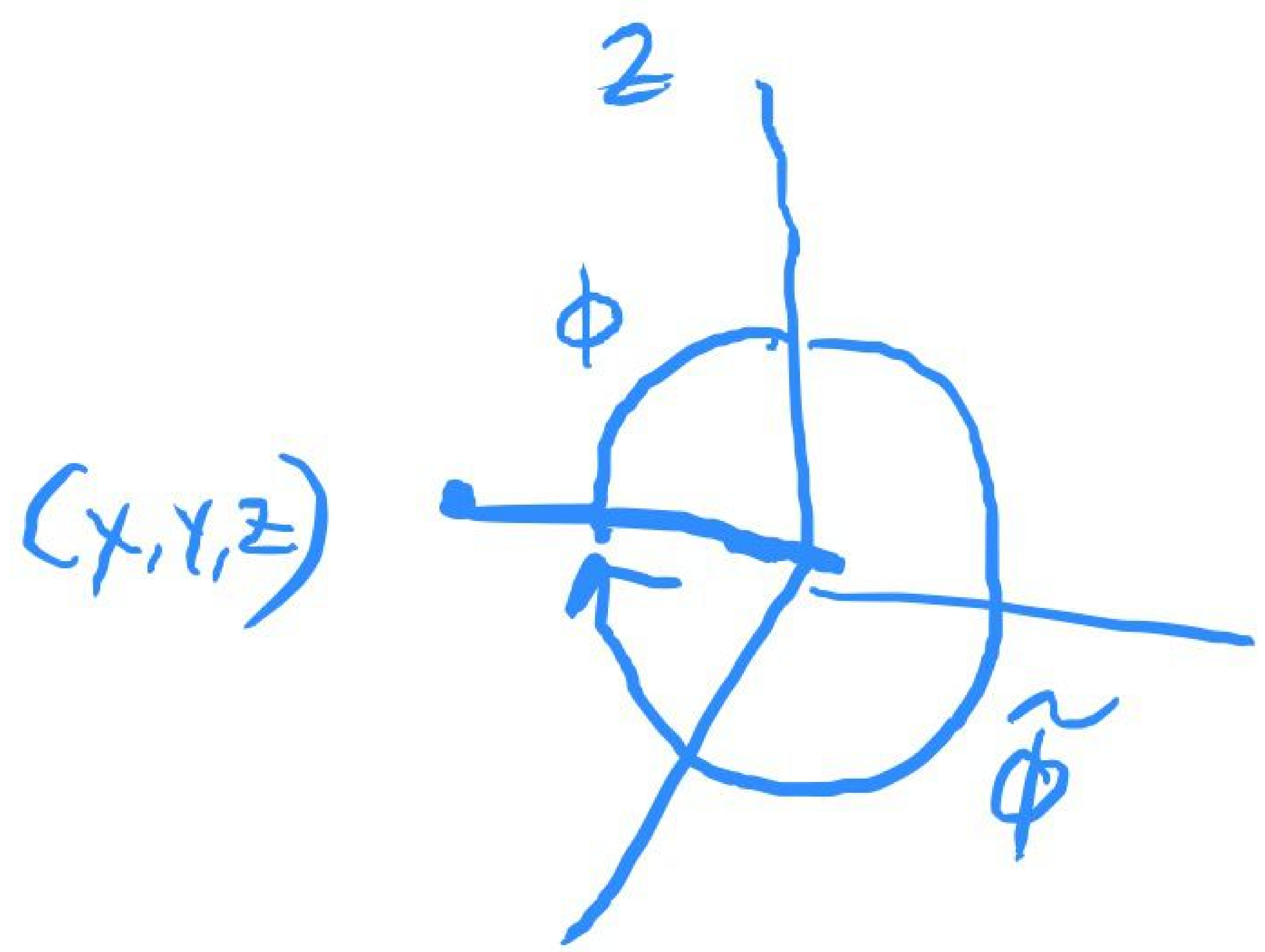
area(line) $= 0$

$\Rightarrow$ lack of 1-1 irrelevant for surface integrals

② what about $D = [0, 2\pi) \times [0, 2\pi)$

i.e. $0 \le \phi \le 2\pi$

$0 \le \theta \le 2\pi$



if we allow $\tilde{\phi} > \pi$

If $(x,y,z) = \Phi(\tilde{\phi}, \tilde{\theta})$

we can also express it as

$$(x,y,z) = \Phi(\phi, \theta)$$

check for yourself

$$\Phi(\phi, \theta) = \Phi(2\pi - \phi, \theta + \pi)$$

(use: $\cos(2\pi - \phi) = \cos\phi$

$\sin(2\pi - \phi) = -\sin\phi$

$\sin(\theta + \pi) = -\sin\theta, \quad \cos(\theta + \pi) = -\cos\theta$)

important point: every point (x,y,z) in S

is image of at least two points in D

<u>check</u>: If we take $D = [0,2\pi] \times [0,2\pi]$

we would get $\iint_D 1 \cdot d\phi \, d\theta = 8\pi a^2$

$=$ twice area of sphere of radius a

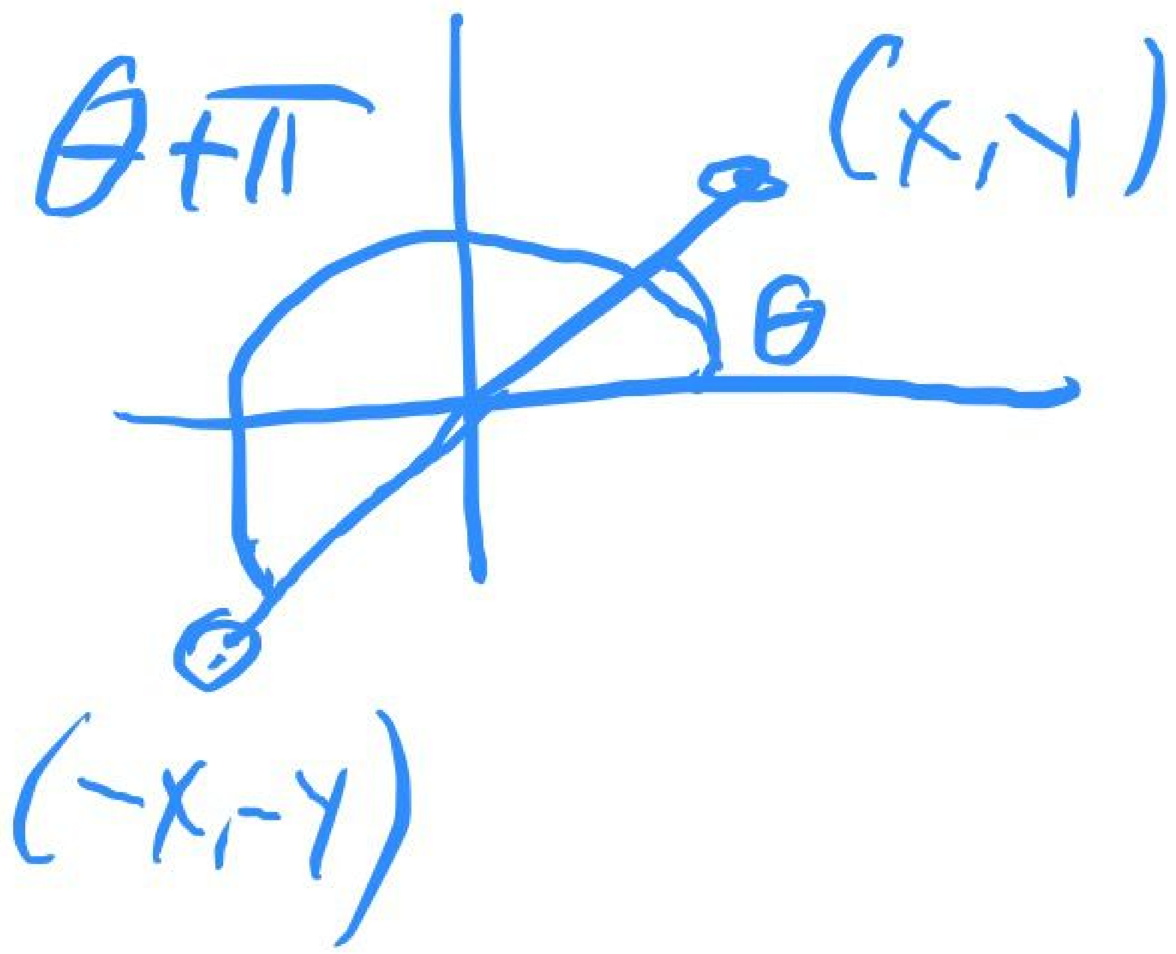


illustration for

$-x = \cos(\theta+\pi) = -\cos\theta = -x$

- surface integral for function $f(x,y,z)$

$$\iint_S f\,dS = \iint_D f(\Phi(u,v))\,\|T_u \times T_v\|\,du\,dv$$

$$T_u = \frac{\partial \Phi}{\partial u}, \quad T_v = \frac{\partial \Phi}{\partial v}$$

- surface integral for vector field F

$$\boxed{\iint_S F \cdot dS = \iint_D F(\Phi(u,v)) \cdot (T_u \times T_v)\,du\,dv}$$

where $T_u \times T_v$ points to specified positive of S

Let $\vec{n} = \vec{n}(x,y,z)$ be the normal vector at $(x,y,z) \in S$ of length 1 pointing to positive side

$$\boxed{\vec{n} = \frac{T_u \times T_v}{\|T_u \times T_v\|}}$$

$$\Rightarrow \iint_S F\cdot dS = \iint_D F(\Phi(u,v))\cdot \underbrace{\frac{T_u\times T_v}{\|T_u\times T_v\|}}_{\vec{n}} \|T_u\times T_v\|\, du\,dv$$

$$\boxed{\iint_S F\cdot dS = \iint_S (F\cdot\vec{n})\, dS}$$

This formula can be useful in certain geometric situations

Recall from last class:

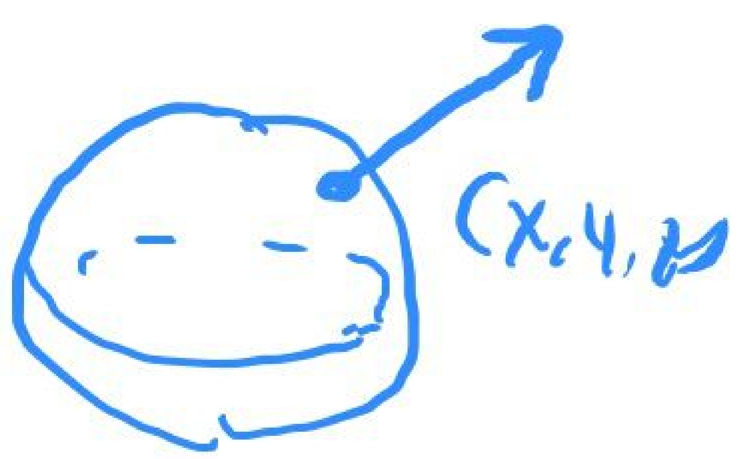
$$F = \frac{\vec{r}}{r^3} \qquad \vec{r}(x,y,z) = (x,y,z)$$

$$r = \|\vec{r}(x,y,z)\| = \sqrt{x^2+y^2+z^2}$$

checked last time: $\boxed{\operatorname{div} F = 0}$

Calculate $\iint_S F \cdot dS$ directly for S = sphere of radius a

using formula from last page



sphere of radius a

$\Rightarrow \|(x,y,z)\| = a$

$$\vec{n} = \frac{1}{a}(x,y,z) = \frac{1}{a}\vec{r}$$

$$\Rightarrow \iint_S F \cdot dS = \iint_S \frac{\vec{r}}{r^3} \cdot \frac{\vec{r}}{a}\, dS$$

(observe: $\vec{r} \cdot \vec{r} = r^2 = a^2$ (on sphere of radius a))

$$= \iint_S \frac{a^2}{a^3 \cdot a}\, dS = \iint \frac{1}{a^2}\, dS = \frac{1}{a^2}\,\text{area}(S) = \frac{4\pi a^2}{a^2} = \boxed{4\pi}$$

$$\Rightarrow \quad \iint_S F \cdot dS = 4\pi \neq 0 = \iiint_B \underbrace{\operatorname{div} F}_{=0} \, dV$$

B = ball of radius a

$\Rightarrow$ Gauss' divergence theorem does NOT hold

reason: F not well-defined at $(0,0,0)$

But the following result holds:

<u>Theorem (Gauss' Law)</u>

$W \subset \mathbb{R}^3$ region with boundary ∂W,

$$\boxed{\iint_{\partial W} \frac{\vec{r}}{r^3} \cdot dS = \begin{cases} 0 & \text{if } \vec{0} \notin W \\ 4\pi & \text{if } \vec{0} \in W \end{cases}}$$

Justification of theorem:

(a) If $\vec{0} \notin W$

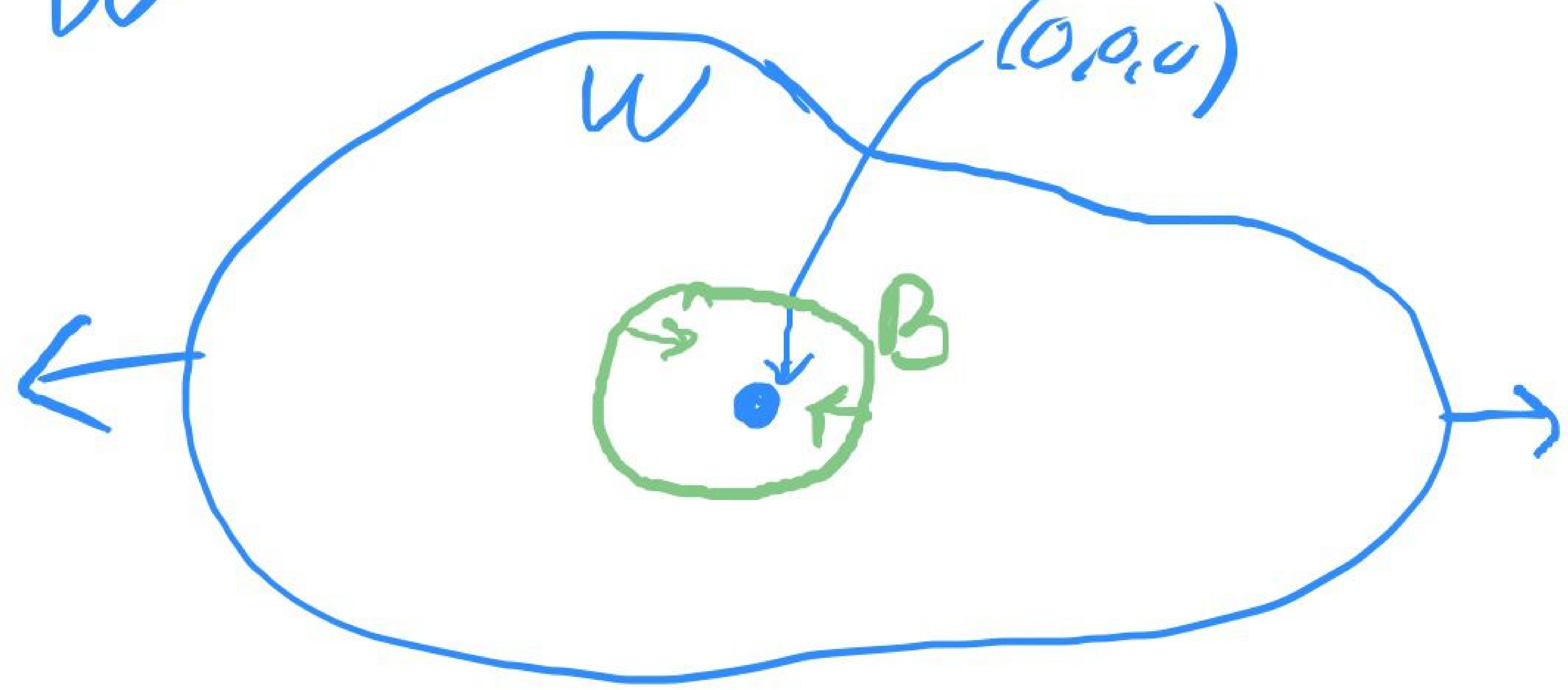
$\Rightarrow$ $F = \frac{\vec{r}}{r^3}$ well-defined in W

$\Rightarrow$ can use divergence theorem

$$\iint_S F \cdot dS = \iiint \operatorname{div} F \, dV = 0$$

$$F = \frac{\vec{r}}{r^3}$$

(b) If $0 \in W$



B ball of radius a such that $B \subset W$

$\Rightarrow$ divergence theorem holds for

$W \setminus B$ (i.e. W w/o ball B)

$$\partial(W \setminus B) = \partial W \cup \partial B$$

normal vector points to outside of $W \setminus B$

$\rightarrow$ points to outside of W

or inside of B

$$\Rightarrow 0 = \iiint_{W \setminus B} \operatorname{div} F \, dV = \iint_{\partial W} F \cdot dS + \iint_{\partial B} F \cdot dS$$

normal vector pointing inside!

$$= \iint_{\partial W} F \cdot dS - \iint_{\partial B} F \cdot dS = \iint_{\partial W} F \cdot dS - 4\pi$$

vector outside of ball

$$\Rightarrow \iint_{\partial W} F \cdot dS = 4\pi$$

$$F = \frac{\vec{r}}{r^3}$$

<u>Remark</u> In electro engineering this result means

The flux through ∂W for a point charge is given by
$$\begin{cases} 4\pi & \text{if point charge inside } W \\ 0 & \text{" " outside } W \end{cases}$$

Problems from Section 8.4 will be posted this afternoon